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Journal of Approximation Theory 127 (2004) 39–60
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

(C, α) summability of Walsh–Kaczmarz–Fourier series ☆

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Received 30 December 2002; accepted in revised form 28 February 2004

Communicated by András Kroó

Dedicated to Professor F. Schipp on the occasion of his 65th birthday

Abstract

The Walsh system will be investigated in the Kaczmarz rearrangement. In an earlier paper we have shown that the maximal operator of the (C,1)-means of the Walsh–Kaczmarz–Fourier series is bounded from the dyadic Hardy space H^p into L^p for every $\frac{1}{2}<p\leqslant 1$. In the present work, we extend this result to the (C, α) means when $0<\alpha\leqslant 1$ and prove their maximal operator $\sigma^\alpha : H^p \to L^p$ is bounded for all $1/(\alpha+1)<p\leqslant 1$. By known results on interpolation we get from this theorem that σ^α is of weak type (1,1) and bounded from L^q into L^q if $1<q\leqslant\infty$. Moreover, the (C, α) means of an integrable function f converge to f a.e.

Keywords: Walsh functions; Cesaro summation; Hardy spaces; Interpolation

1. Introduction

The first result on the a.e. convergence of the (C,1) means of Walsh–Fourier series is due to Fine [1], if the Walsh functions are considered by Paley's ordering. This result follows also from a maximal inequality of Schipp [4]. Namely, Schipp proved that the maximal operator σ^1_{WP} of the (C,1) summation with respect to the Walsh–Paley Fourier–series is of weak type (1,1) and bounded from L^p to L^p when

☆This research was supported by the Hungarian Research Fund OTKA T032719.
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0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.02.003

$1<p\leqslant\infty$. In the case $p=1$ this boundedness is not true but σ^1_{WP} is bounded as a map from the dyadic Hardy space H^1 to L^1 (see [2] or [6]). Later some extensions are proved by Weisz [10], for example that $\sigma^1_{\mathrm{WP}}: H^p\to L^p$ is bounded for every $\frac{1}{2}<p\leqslant 1$. A counterexample shows that this result fails to hold for $0<p<1/2$ (see [8]).

The so-called Walsh–Kaczmarz system was also investigated by many authors. Thus the Kaczmarz analogue of Schipp's results was given by Gát [3]. Moreover, he proved also an (H^1,L^1)-like inequality $\|\sigma^1_{\mathrm{WK}}f\|_1\leqslant |||f|||_{H^1}$ $(f\in H^1)$. In [7] we have shown that the theorem of Weisz mentioned above remains true for σ^1_{WK} instead of σ^1_{WP}.

The maximal operator $\sigma^\alpha_{\mathrm{WP}}$ $(0<\alpha\leqslant 1)$ of the (C,α) means of the Walsh–Paley Fourier-series was investigated by Weisz [11]. In his paper Weisz proved the boundedness of $\sigma^\alpha_{\mathrm{WP}}: H^p\to L^p$ when $1/(\alpha+1)<p\leqslant 1$. In the present work the exact analogue of this statement will be shown for $\sigma^\alpha_{\mathrm{WK}}$, i.e. for the maximal operator of the Walsh–Kaczmarz (C,α) summation. To this end we prove also the uniform L^1-boundedness of the Walsh–Kaczmarz (C,α) kernels which implies evidently that $\sigma^\alpha_{\mathrm{WK}}: L^\infty\to L^\infty$ is bounded. We remark that known theorems on interpolation imply the weak typeness (1,1) of $\sigma^\alpha_{\mathrm{WK}}$ and the boundedness of $\sigma^\alpha_{\mathrm{WK}}: L^s\to L^s$ $(1<s\leqslant\infty)$. Moreover, by standard density argument the a.e. convergence of the Walsh–Kaczmarz (C,α) means follows for every integrable function. This is an extension of a theorem on the (C,1) summation of Gát [3].

2. Definitions and notations

The Walsh–Paley system is a special product system generated by the so-called Rademacher functions r_n $(n\in\mathbf{N}:=\{0,1,\dots\})$. To their definition let r be the function given on the interval $[0,1)$ by

$$r(x):=\begin{cases}1 & (0\leqslant x<1/2),\\ -1 & (1/2\leqslant x<1)\end{cases}$$

and extended to the whole real line $\mathbf{R}$ periodically by 1. Now, define $r_n(x):=r(2^n x)$ $(x\in[0,1], n\in\mathbf{N})$. Then the usual product system $(w_n, n\in\mathbf{N})$ of r_n's is obtained in the following way:

$$w_n:=\prod_{k=0}^{\infty} r_k^{n_k}\qquad (n\in\mathbf{N}),$$

where $n=\sum_{k=0}^{\infty} n_k 2^k$ is the binary decomposition of n, i.e. $n_k\in\{0,1\}(k\in\mathbf{N})$. It is well-known (for details see the book [5]) that $(w_n, n\in\mathbf{N})$ is a complete orthonormal system with respect to the Lebesgue measure of $[0,1]$. Denote $D_k:=\sum_{j=0}^{k-1} w_j$ $(k\in\mathbf{N}, D_0:=0)$ the kth Walsh–Dirichlet kernel. Then a basic property of the Walsh functions is

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \leqslant x < 2^{-n}) \\ & \\ 0 & (2^{-n} \leqslant x < 1) \end{cases} \qquad (n \in \mathbf{N}). \tag{1}$$

The interval $[0,1)$ can be treated as the so-called dyadic group, i.e. the set of all 0–1 sequences $(x_k, k \in \mathbf{N})$ $(x_k \in \{0,1\}$ $(k \in \mathbf{N}))$. The group operation $\dot{+}$ is the coordinate-wise addition modulo 2, i.e. if $x = (x_k, k \in \mathbf{N})$, $y = (y_k, k \in \mathbf{N})$ then $x \dot{+} y := (x_k \oplus y_k, k \in \mathbf{N})$, where $a \oplus b$ denotes the addition modulo 2 of $a, b \in \mathbf{N}$. For example, the Rademacher functions can be computed in this sense as $r_n(x) = (-1)^{x_n}$ $(x \in [0,1), n \in \mathbf{N})$. Furthermore, $D_{2^n} = 2^n \chi_{I_n}$ $(n \in \mathbf{N})$ where I_n is the set of all $(x_k, k \in \mathbf{N})$ such that $x_0 = x_1 = \cdots = x_{n-1} = 0$ and χ_{I_n} is its characteristic function.

The Walsh–Fejér kernels K_n $(0 < n \in \mathbf{N})$ play also an important part in our investigations:

$$K_n := \frac{1}{n} \sum_{j=1}^{n} D_j = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) w_k.$$

Let $K_0 := 0$. The next estimation with respect to K_n's will be used often in this work: if $x \in [0,1), 0 < n \in \mathbf{N}$ then

$$|K_n(x)| \leqslant \sum_{j=0}^{s} 2^{j-s-1} \sum_{i=j}^{s} (D_{2^i}(x) + D_{2^i}(x \dot{+} 2^{-j-1})) \quad (2^s \leqslant n < 2^{s+1}). \tag{2}$$

From this it follows by (1) the uniform L^1-boundedness of K_n's:

$$\sup_n \|K_n\|_1 < \infty. \tag{3}$$

In this work the so-called Kaczmarz rearrangement $(\Psi_n, n \in \mathbf{N})$ (called *Walsh–Kaczmarz system*) of $(w_n, n \in \mathbf{N})$ will be investigated, where Ψ_n's are defined in the following way. If $0 < n \in \mathbf{N}$ then there is a unique $s \in \mathbf{N}$ such that the binary representation of n is of the form $n = 2^s + \sum_{k=0}^{s-1} n_k 2^k$. Then let

$$\Psi_n := r_s \prod_{k=0}^{s-1} r_{s-k-1}^{n_k}$$

and $\Psi_0 := w_0$. It is not hard to see that $\Psi_{2^m} = w_{2^m} = r_m$ and $\{\Psi_k : k = 2^m, \ldots, 2^{m+1} - 1\} = \{w_k : k = 2^m, \ldots, 2^{m+1} - 1\}$ $(m \in \mathbf{N})$. Furthermore, if

$$\tau_s(x) := (x_{s-1}, x_{s-2}, \ldots, x_1, x_0, x_s, x_{s+1}, \ldots) \quad (x \in [0,1))$$

then

$$\Psi_n(x) = w_n(\tau_s(x)) = r_s(x) w_{n-2^s}(\tau_s(x)). \tag{4}$$

It is clear that $(\Psi_n, n \in \mathbf{N})$ is also a complete orthonormal system and $D_{2^j}(\tau_j(x)) = D_{2^j}(x)$ $(j \in \mathbf{N}, x \in [0,1))$.

Let $0<\alpha\leqslant 1, k\in\mathbf{N}$ and

$$A_k^\alpha := \prod_{i=1}^{k} \frac{\alpha+i}{i}.$$

Then the nth (C,α) *Walsh–Kaczmarz Fejér kernel* with respect to Ψ_k's will be denoted by

$$\mathscr{K}_n^\alpha := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha \Psi_k \quad (0<n\in\mathbf{N}).$$

Furthermore, let

$$\sigma_n^\alpha f(x) := \int_0^1 f(t)\mathscr{K}_n^\alpha(x\dot{+}t)\,dt \quad (x\in[0,1), n\in\mathbf{N})$$

the nth (C,α) *Walsh–Kaczmarz Fejér mean* of $f\in L^1[0,1)$. The next maximal operator will be investigated in the further sections:

$$\sigma^\alpha f := \sup_n |\sigma_n^\alpha f|.$$

We remark (see e.g. [14]) that $A_k^\alpha \sim O(k^\alpha)$ $(k\to\infty)$.

3. Hardy spaces

Here, we give only the most important concepts with respect to the dyadic Hardy spaces. (For details see, e.g. the books of [9,12].) To this end let the *maximal function* of $f\in L^1[0,1)$ be given by

$$f^*(x) = \sup_n 2^n \left| \int_{x\dot{+}I_n} f \right| \quad (x\in[0,1))$$

and for $0<p<\infty$ denote $H^p[0,1)$ the space of f's for which $\|f\|_{H^p} := \|f^*\|_p < \infty$.

A function $a\in L^\infty[0,1)$ is called a *p-atom* if either a is identically equal to 1 or if there exists a dyadic interval $I = x\dot{+}I_N$ for some $N\in\mathbf{N}$, $x\in[0,1)$ such that

$$\text{supp } a\subset I, \|a\|_\infty \leqslant 2^{N/p} \quad \text{and} \quad \int_0^1 a = 0.$$

We shall say that a is *supported* on I.

A sublinear operator T which maps $H^p[0,1)$ into the collection of measurable functions defined on $[0,1)$ will be called *p-quasi-local* if there exists a constant C_p such that

$$\int_{[0,1)\setminus I} |Ta|^p \leqslant C_p \tag{5}$$

for every p-atom a supported on I. (Here and later C_p, C will denote positive constants depending at most on p and α, although not always the same in different occurrences.) Assume the L^∞-boundedness of T, i.e. that

$\|Tf\|_\infty \leqslant C\|f\|_\infty$ $(f \in L^\infty[0,1))$. Then it is known that for T to be bounded from $H^p[0,1)$ to $L^p[0,1)$ it is sufficient that T is p-quasi-local.

4. Cesaro summability

First of all we show the analogue of (3) for $\mathscr{K}_n^\alpha$'s, i.e. that the $\mathscr{K}_n^\alpha$ $(n \in \mathbf{N})$ kernels are uniformly L^1-bounded. In other words the following theorem holds:

Theorem 1. *For all* $0<\alpha\leqslant 1$ *we have*

$$\sup_n \|\mathscr{K}_n^\alpha\|_1 < \infty.$$

This statement implies evidently the next corollary:

Corollary 1. *The maximal operator* σ^α $(0<\alpha\leqslant 1)$ *is of type* (∞,∞), *i.e.*

$$\|\sigma^\alpha f\|_\infty \leqslant C\|f\|_\infty \quad (f \in L^\infty[0,1)).$$

Further we deal with the (H^p, L^p)-boundedness of σ^α. We remember the special case $\alpha = 1$, i.e. (see [7]) that $\sigma^1 : H^p[0,1) \to L^p[0,1)$ is bounded when $p > 1/2$. Now, we extend this theorem to $0<\alpha<1$ and show

Theorem 2. *Let* $0<\alpha\leqslant 1$ *and* $1/(\alpha+1)<p\leqslant 1$. *Then*

$$\|\sigma^\alpha f\|_p \leqslant C\|f\|_{H^p} \quad (f \in H^p[0,1)).$$

Applying known results on interpolation (see e.g. the books [9,12]) we get

Corollary 2. *For every* $0<\alpha\leqslant 1$ *and* $1<p\leqslant\infty$ *the maximal operator* $\sigma^\alpha : L^p[0,1)\to L^p[0,1)$ *is bounded. Moreover,* σ^α *is of weak type* (1,1) *and* $\sigma_n^\alpha f \to f$ $(n\to\infty)$ *a.e. if* $f \in L^1[0,1)$.

5. Proof of theorems

Theorem 1 is a direct consequence of (3) if $\alpha = 1$ (see e.g. [7]). Hence it can be assumed that $0<\alpha<1$.

Let $n = \sum_{k=1}^q 2^{n_k}$ be the binary decomposition of $0<n\in\mathbf{N}$, where $n_k \in \mathbf{N}$ $(k = 1,\dots,q)$ and $n_k > n_{k+1}$ $(k = 1,\dots,q-1)$. Then $\mathscr{K}_n^\alpha$ can be written as the sum $\mathscr{K}_{n1}^\alpha + \mathscr{K}_{n2}^\alpha$ with

$$\mathscr{K}_{n1}^\alpha := \frac{1}{A_{n-1}^\alpha}\sum_{k=0}^{2^{n_1}-1} A_{n-k-1}^\alpha \Psi_k, \quad \mathscr{K}_{n2}^\alpha := \frac{1}{A_{n-1}^\alpha}\sum_{k=2^{n_1}}^{n-1} A_{n-k-1}^\alpha \Psi_k. \tag{6}$$

Applying (4) we get for $x \in [0,1)$

$$\begin{aligned}\mathscr{K}^{\alpha}_{n1}(x) &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A^{\alpha}_{n-1-(2^{j+1}-1-k)} \Psi_{2^{j+1}-1-k}(x) \\ &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A^{\alpha}_{n-2^{j+1}+k} w_{2^{j+1}-1-k}(\tau_j(x)) \\ &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A^{\alpha}_{n-2^{j+1}+k} w_{2^{j+1}-1}(\tau_j(x)) w_k(\tau_j(x)) \\ &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=0}^{2^j-1} A^{\alpha}_{n-2^{j+1}+k} (D_{k+1}(\tau_j(x)) - D_k(\tau_j(x))).\end{aligned}$$

Therefore by Abel transformation it follows that

$$\begin{aligned}\mathscr{K}^{\alpha}_{n1}(x) &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \left(\sum_{k=1}^{2^j} A^{\alpha}_{n-2^{j+1}+k-1} D_k(\tau_j(x)) \right. \\ &\quad \left. - \sum_{k=0}^{2^j-1} A^{\alpha}_{n-2^{j+1}+k} D_k(\tau_j(x)) \right) \\ &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) A^{\alpha}_{n-2^j-1} D_{2^j}(\tau_j(x)) \\ &\quad + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} (A^{\alpha}_{n-2^{j+1}+k-1} - A^{\alpha}_{n-2^{j+1}+k}) D_k(\tau_j(x)) \\ &= 1 + \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) A^{\alpha}_{n-2^j-1} D_{2^j}(\tau_j(x)) \\ &\quad - \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A^{\alpha-1}_{n-2^{j+1}+k} D_k(\tau_j(x)) =: \mathscr{K}^{\alpha 1}_{n1} + \mathscr{K}^{\alpha 2}_{n1}.\end{aligned}$$

Taking into consideration $D_k = kK_k - (k-1)K_{k-1}$ $(0 < k \in \mathbf{N})$ we can transform $\mathscr{K}^{\alpha 2}_{n1}$ as follows:

$$\begin{aligned}\mathscr{K}^{\alpha 2}_{n1}(x) &= \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A^{\alpha-1}_{n-2^{j+1}+k} \\ &\quad \times (kK_k(\tau_j(x)) - (k-1)K_{k-1}(\tau_j(x))) \\ &= \frac{1}{A^{\alpha}_{n-1}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \left(\sum_{k=1}^{2^j-1} A^{\alpha-1}_{n-2^{j+1}+k} (kK_k(\tau_j(x)) \right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k=0}^{2^j-2} A_{n-2^{j+1}+k+1}^{\alpha-1} k K_k(\tau_j(x))) \Bigg) \\
&= \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x))(2^j-1) A_{n-2^j-1}^{\alpha-1} K_{2^{j-1}}(\tau_j(x)) \\
&\quad - \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-2} k A_{n-2^{j+1}+k+1}^{\alpha-2} K_k(\tau_j(x)) \\
&=: A(x) + B(x).
\end{aligned}
$$

Now, if we apply estimation (2) it follows for $B(x)$ that

$$
\begin{aligned}
|B(x)| &\leqslant Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{s=1}^{j-1} \sum_{l=2^{s-1}}^{2^s-1} |A_{n-2^j+l+1}^{\alpha-2}| \sum_{i=0}^{s-1} \sum_{m=0}^{i} 2^m \\
&\quad \times (D_{2^i}(\tau_{j-1}(x)) + D_{2^i}(\tau_{j-1}(x) \dot{+} e_m)) \\
&\leqslant Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} \sum_{s=i+1}^{j-1} \sum_{l=2^{s-1}}^{2^s-1} (n-2^j+l)^{\alpha-2} \sum_{m=0}^{i} 2^m \\
&\quad \times (D_{2^i}(\tau_{j-1}(x)) + D_{2^i}(\tau_{j-1}(x) \dot{+} e_m)) \\
&= Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} \alpha_{ij} \sum_{m=0}^{i} 2^m (D_{2^i}(\tau_{j-1}(x)) + D_{2^i}(\tau_{j-1}(x) \dot{+} e_m)) \\
&\leqslant Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} \alpha_{ij} \left(2^i D_{2^i}(\tau_{j-1}(x)) + \sum_{m=0}^{i-1} 2^m D_{2^i}(\tau_{j-1}(x) \dot{+} e_m) \right),
\end{aligned}
$$

where $e_m \coloneqq 2^{-m-1} = (0, \dots, 0, 1, 0, \cdots)$ and

$$
\alpha_{ij} \coloneqq \sum_{s=i+1}^{j-1} \sum_{l=2^{s-1}}^{2^s-1} (n-2^j+l)^{\alpha-2} \leqslant C \int_{2^i}^{2^{j-1}} (n-2^j+x)^{\alpha-2}\, dx \leqslant C 2^{i(\alpha-1)}.
$$

An analogous calculation shows that the same estimation holds for $A(x)$ instead of $B(x)$. Thus $\mathscr{K}_{n1}^{\alpha 2}(x)$ can be estimated as

$$
|\mathscr{K}_{n1}^{\alpha 2}(x)| \leqslant Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{i(\alpha-1)} \left(2^i D_{2^i}(\tau_{j-1}(x)) + \sum_{m=0}^{i-1} 2^m D_{2^i}(\tau_{j-1}(x) \dot{+} e_m) \right).
$$

Applying (1) the previous estimation implies for $||\mathscr{K}_{n1}^{\alpha 2}||_1$ that

$$
||\mathscr{K}_{n1}^{\alpha 2}||_1 \leqslant Cn^{-\alpha} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} 2^i \leqslant Cn^{-\alpha} \sum_{j=1}^{n_1-1} \sum_{i=0}^{j-2} 2^{i\alpha} \leqslant C \frac{2^{n_1 \alpha}}{n^{\alpha}} \leqslant C.
$$

The L^1-norm estimation of $\mathscr{K}_{n1}^{\alpha 1}$ is very simple. Indeed, taking into account $w_{2^{j+1}-1}(\tau_j(x)) = r_j(x)$ when $x_0 = \cdots = x_{j-1} = 0$ it follows by (1) that

$$w_{2^{j+1}-1}(\tau_j(x))D_{2^j}(x) = r_j(x)D_{2^j}(x) = D_{2^{j+1}}(x) - D_{2^j}(x),$$

i.e.

$$\begin{aligned}
\mathscr{K}_{n1}^{\alpha 1}(x) &= 1 + \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha}(D_{2^{j+1}}(x) - D_{2^j}(x)) \\
&= 1 + \frac{1}{A_{n-1}^{\alpha}} \left(\sum_{j=1}^{n_1} A_{n-2^{j-1}-1}^{\alpha} D_{2^j}(x) - \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha} D_{2^j}(x) \right) \\
&= 1 + \frac{1}{A_{n-1}^{\alpha}} A_{n-2^{n_1-1}-1}^{\alpha} D_{2^{n_1}}(x) - \frac{1}{A_{n-1}^{\alpha}} A_{n-2}^{\alpha} \\
&\quad + \frac{1}{A_{n-1}^{\alpha}} \sum_{j=1}^{n_1-1} (A_{n-2^{j-1}-1}^{\alpha} - A_{n-2^j-1}^{\alpha}) D_{2^j}(x)
\end{aligned}$$

from which by (1)

$$\begin{aligned}
\|\mathscr{K}_{n1}^{\alpha 1}\|_1 &\leqslant C + \frac{1}{A_{n-1}^{\alpha}} \sum_{j=1}^{n_1-1} (A_{n-2^{j-1}-1}^{\alpha} - A_{n-2^j-1}^{\alpha}) \\
&= C + \frac{1}{A_{n-1}^{\alpha}} (A_{n-2}^{\alpha} - A_{n-2^{n_1-1}-1}^{\alpha}) \leqslant C
\end{aligned}$$

follows.

Now, we investigate $\mathscr{K}_{n2}^{\alpha}(x)$ $(x \in [0,1))$ (for the idee see [13] or [11,12]):

$$\begin{aligned}
\mathscr{K}_{n2}^{\alpha}(x) &= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2^{n_1}}^{n-1} A_{n-k-1}^{\alpha} \Psi_k(x) = \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{q-1} \sum_{j=2^{n_1}+\cdots+2^{n_k}}^{2^{n_1}+\cdots+2^{n_{k+1}}-1} A_{n-j-1}^{\alpha} \Psi_j(x) \\
&= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{q-1} \sum_{j=0}^{2^{n_{k+1}}-1} A_{n-1-(2^{n_1}+\cdots+2^{n_{k+1}}-1-j)}^{\alpha} \Psi_{2^{n_1}+\cdots+2^{n_{k+1}}-1-j}(x) \\
&= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{q-1} w_{2^{n_1}+\cdots+2^{n_{k+1}}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_{k+1}}-1} A_{n-(2^{n_1}+\cdots+2^{n_{k+1}})+j}^{\alpha} w_j(\tau_{n_1}(x)) \\
&= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{2^{n_1}+\cdots+2^{n_k}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n-(2^{n_1}+\cdots+2^{n_k})+j}^{\alpha} w_j(\tau_{n_1}(x)) \\
&= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha} w_j(\tau_{n_1}(x)),
\end{aligned}$$

where $n^{(k)} := n - \sum_{i=1}^{k} 2^{n_i}$ $(k = 1, \ldots, q)$. By means of Abel transformation this yields >

$$\begin{aligned}
\mathscr{K}_{n2}^{\alpha}(x) =& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha}(D_{j+1}(\tau_{n_1}(x)) - D_j(\tau_{n_1}(x)))\\
=& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \Bigg(\sum_{j=1}^{2^{n_k}} A_{n^{(k)}+j-1}^{\alpha} D_j(\tau_{n_1}(x))\\
&- \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha} D_j(\tau_{n_1}(x)) \Bigg)\\
=& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \Big(A_{n^{(k)}+2^{n_k}-1}^{\alpha} D_{2^{n_k}}(\tau_{n_1}(x))\\
&- \sum_{j=1}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha-1} D_j(\tau_{n_1}(x)) \Bigg)\\
=& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha} D_{2^{n_k}}(\tau_{n_1}(x))\\
&- \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=1}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha-1}(jK_j(\tau_{n_1}(x))\\
&- (j-1)K_{j-1}(\tau_{n_1}(x)))\\
=& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha} D_{2^{n_k}}(\tau_{n_1}(x))\\
&- \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \Bigg(\sum_{j=1}^{2^{n_k}-1} jA_{n^{(k)}+j}^{\alpha-1} K_j(\tau_{n_1}(x))\\
&- \sum_{j=0}^{2^{n_k}-2} jA_{n^{(k)}+j+1}^{\alpha-1} K_j(\tau_{n_1}(x)) \Bigg)\\
=& \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha} D_{2^{n_k}}(\tau_{n_1}(x))\\
&- \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha-1} (2^{n_k}-1) K_{2^{n_k}-1}(\tau_{n_1}(x))\\
&+ \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=1}^{2^{n_k}-2} jA_{n^{(k)}+j+1}^{\alpha-2} K_j(\tau_{n_1}(x))\\
=:& \mathscr{K}_{n2}^{\alpha 1}(x) + \mathscr{K}_{n2}^{\alpha 2}(x) + \mathscr{K}_{n2}^{\alpha 3}(x).
\end{aligned}$$

First we deal with $||\mathcal{K}_{n2}^{\alpha 1}||_1$:

$$||\mathcal{K}_{n2}^{\alpha 1}||_1 \leqslant Cn^{-\alpha} \sum_{k=2}^{q} A^{\alpha}_{n^{(k-1)}-1} \leqslant Cn^{-\alpha} \sum_{k=2}^{q} (n^{(k-1)})^{\alpha}$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{q} 2^{\alpha n_k} \leqslant Cn^{-\alpha} 2^{\alpha n_2} \leqslant C.$$

Now, we investigate $||\mathcal{K}_{n2}^{\alpha 2}||_1$:

$$||\mathcal{K}_{n2}^{\alpha 2}||_1 \leqslant Cn^{-\alpha} \sum_{k=2}^{q} 2^{n_k} (n^{(k-1)})^{\alpha-1} ||K_{2^{n_k}-1}||_1$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{q} 2^{n_k} 2^{(\alpha-1)n_k} \leqslant Cn^{-\alpha} 2^{\alpha n_2} \leqslant C.$$

Finally, $||\mathcal{K}_{n2}^{\alpha 3}||_1$ can be estimated by (3) as follows:

$$||\mathcal{K}_{n2}^{\alpha 3}||_1 \leqslant Cn^{-\alpha} \sum_{k=2}^{r} \sum_{j=1}^{2^{n_k}-2} j|A^{\alpha-2}_{n^{(k)}+j+1}|||K_j||_1$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} \sum_{j=2^l}^{2^{l+1}-1} j(n^{(k)}+j)^{\alpha-2}$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} (n^{(k)}+2^l)^{\alpha-2} \sum_{j=2^l}^{2^{l+1}-1} j$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} (n^{(k)}+2^l)^{\alpha-2} 2^{2l} \leqslant Cn^{-\alpha} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} 2^{(\alpha-2)l} 2^{2l}$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{r} 2^{\alpha n_k} \leqslant C.$$

This proves Theorem 1. □

Proof of Theorem 2. For the special case $\alpha = 1$ see [7]. Thus let $0 < \alpha < 1$ and $1/(\alpha+1) < p \leqslant 1$. We need to show that σ^{α} is p-quasi-local. To this end let a be a p-atom supported on I_N for some $N \in \mathbf{N}$. Then for all $n \in \mathbf{N}$ the assumption $n \leqslant 2^N$ implies $\sigma_n^{\alpha} a = 0$, therefore $\sigma^{\alpha} a = \sup_{n, n > 2^N} |\sigma_n^{\alpha} a|$. For $n \in \mathbf{N}$, $n > 2^N$ we use the decomposition $n = \sum_{k=1}^{q} 2^{n_k}$ with $n_k \in \mathbf{N}$ $(k = 1, \dots, q)$ and $n_k > n_{k+1}$ $(k = 1, \dots, q-1)$. Furthermore, we can assume $n_1 \geqslant N$. As in the proof of Theorem 1 (see (6)) let $\mathcal{K}_n^{\alpha}$ be decomposed as $\mathcal{K}_n^{\alpha} = \mathcal{K}_{n1}^{\alpha} + \mathcal{K}_{n2}^{\alpha}$, i.e. $\sigma_n^{\alpha} a = \sigma_{n1}^{\alpha} a + \sigma_{n2}^{\alpha} a$ where for $x \in [0,1)$

$$\sigma_{n1}^{\alpha} a(x) := \int_0^1 a(t)\mathscr{K}_{n1}^{\alpha}(x \dot{+} t)\,dt, \quad \sigma_{n2}^{\alpha} a(x) := \int_0^1 a(t)\mathscr{K}_{n2}^{\alpha}(x \dot{+} t)\,dt.$$

Here (see again the proof of Theorem 1) $\mathscr{K}_{n1}^{\alpha} = \mathscr{K}_{n1}^{\alpha 1} + \mathscr{K}_{n1}^{\alpha 2}$ which implies the decomposition $\sigma_{n1}^{\alpha} a = \sigma_{n1}^{\alpha 1} a + \sigma_{n1}^{\alpha 2} a$ by means

$$\sigma_{n1}^{\alpha 1} a(x) := \int_0^1 a(t)\mathscr{K}_{n1}^{\alpha 1}(x \dot{+} t)\,dt, \quad \sigma_{n1}^{\alpha 2} a(x) := \int_0^1 a(t)\mathscr{K}_{n1}^{\alpha 2}(x \dot{+} t)\,dt \quad (x \in [0,1)).$$

If $x \in [0,1)\backslash I_N$ then by (1) $\int_0^1 a(t) D_j(x \dot{+} t)\,dt = 0$ holds for all $j = 0, \ldots, 2^{n_1}$. Therefore (see the proof of Theorem 1 for $\mathscr{K}_{n1}^{\alpha 1}$) we get $\sigma_{n1}^{\alpha 1} a(x) = 0$.

Now, we consider $\sigma_{n1}^{\alpha 2}(x)$ for $x \in [0,1)\backslash I_N$. Taking into account the estimation for $\mathscr{K}_{n1}^{\alpha 2}(x)$ (see the proof of Theorem 1) we get

$$\begin{aligned}|\sigma_{n1}^{\alpha 2}(x)| &\leqslant C n^{-\alpha} 2^{N/p} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \Bigg(2^i \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t))\,dt \\ &\quad + \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\,dt \Bigg) \\ &= C n^{-\alpha} 2^{N/p} \left(\sum_{j=1}^{N+1} \cdots + \sum_{j=N+2}^{n_1} \cdots \right) =: C(\delta_{nN}(x) + \Delta_{nN}(x)).\end{aligned}$$

First we investigate $\delta_{nN}(x)$. Since $x \in [0,1)\backslash I_N$ thus there exists a unique $s = 0, \ldots, N-1$ such that $x \in I_s \backslash I_{s+1}$. Decompose $\delta_{nN}(x)$ in the following way: $\delta_{nN}(x) = C n^{-\alpha} 2^{N/p} (\sum_{j=1}^{s+1} \cdots + \sum_{j=s+2}^{N+1} \cdots)$. Then

$$\begin{aligned} n^{-\alpha} 2^{N/p} \sum_{j=1}^{s+1} \cdots &= n^{-\alpha} \sum_{j=1}^{s+1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \Bigg(2^i \int_{I_N} D_{2^i}(x \dot{+} t)\,dt \\ &\quad + \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(x \dot{+} t \dot{+} e_m)\,dt \Bigg),\end{aligned}$$

where by the basic relation (1) $D_{2^i}(x \dot{+} t \dot{+} e_m) = 0$, $D_{2^i}(x \dot{+} t) = 2^i$ for all $t \in I_N$. Therefore

$$\begin{aligned} n^{-\alpha} 2^{N/p} \sum_{j=1}^{s+1} \cdots &= n^{-\alpha} 2^{N/p} \sum_{j=1}^{s+1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} 2^i \int_{I_N} D_{2^i}(x \dot{+} t)\,dt \\ &= n^{-\alpha} 2^{N/p} \sum_{j=1}^{s+1} \sum_{i=0}^{j-2} 2^{(\alpha+1)i} 2^{-N} \leqslant C 2^{N(1/p-\alpha-1)} 2^{(\alpha+1)s}.\end{aligned}$$

To the estimation of $n^{-\alpha} 2^{N/p} \sum_{j=s+2}^{N+1} \cdots$ let the natural numbers $\gamma_0 := s < \gamma_1 < \cdots < \gamma_v \leqslant N-1$ be given, $\gamma := (\gamma_0, \ldots, \gamma_v)$ and define $A_{s\gamma}$ as the set of all $z \in I_s \backslash I_{s+1}$ such that $z_{\gamma_l} = z_{\gamma_l+1} = \cdots = z_{\gamma_{l+1}-1} = 1$ when l is even and $z_{\gamma_l} = z_{\gamma_l+1} =$

$\cdots = z_{\gamma_{l+1}-1} = 0$ when l is odd ($l = 0, \ldots, v$). Then $\bigcup_\gamma A_{s\gamma}$ is a pairwise disjoint decomposition of $I_s \backslash I_{s+1}$ where the measure of each $A_{s\gamma}$ is 2^{-N}. Furthermore, if $x \in A_{s\gamma}$, $j = s+2, \ldots, N+1$ and $i = 0, \ldots, j-2$ then by (1) $D_{2^i}(\tau_{j-1}(x \dot{+} t)) \neq 0$ ($t \in I_N$) iff $j-2 \in [\gamma_l, \gamma_{l+1})$ for some odd $l = 1, \ldots, v$ and $i \leqslant j - \gamma_l - 1$. Thus we can conclude for $x \in A_{s\gamma}$ that

$$\begin{aligned}
n^{-\alpha} 2^{N/p} \sum_{j=s+2}^{N+1} \cdots &\leqslant C n^{-\alpha} 2^{N/p} \sum_{l, l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=0}^{j-\gamma_l-1} 2^{(\alpha-1)i} 2^{2i} 2^{-N} \\
&\quad + C n^{-\alpha} 2^{N/p} \sum_{j=s+2}^{N+1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \\
&\quad \times \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\leqslant C 2^{N(1/p-\alpha-1)} \sum_{l, l \text{ is odd}} 2^{(\alpha+1)(\gamma_{l+1}-\gamma_l)} \\
&\quad + C n^{-\alpha} 2^{N/p} \sum_{j=s+2}^{N+1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \\
&\quad \times \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt.
\end{aligned}$$

Here the last sum can be estimated as

$$\begin{aligned}
& n^{-\alpha} 2^{N/p} \sum_{j=s+2}^{N+1} \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\leqslant C 2^{N(1/p-\alpha)} \sum_{j=s+2}^{N+1} \left(\sum_{i=0}^{j-s-3} \cdots + \sum_{i=j-s-2}^{j-2} \cdots \right) \\
&\leqslant C 2^{N(1/p-\alpha)} \sum_{j=s+2}^{N+1} \left(\sum_{i=0}^{j-s-3} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \right. \\
&\quad \left. + \sum_{i=j-s-2}^{j-2} 2^{(\alpha-1)i} 2^{j-s-3} \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_{j-s-3})\, dt \right) \\
&= C 2^{N(1/p-\alpha)} \left(\sum_{l, l \text{ is even}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=0}^{j-s-3} \cdots + \sum_{l, l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=0}^{j-s-3} \cdots \right) \\
&\quad + C 2^{N(1/p-\alpha)} \sum_{j=s+2}^{N+1} \sum_{i=j-s-2}^{j-2} 2^{(\alpha-1)i} 2^{j-s-3} \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_{j-s-3})\, dt \\
&=: C(x) + D(x).
\end{aligned}$$

Equality (1) implies evidently $D(x) = 0$ if $x \notin I_N(e_s)$. On the other hand, if $x \in I_N(e_s)$ then

$$D(x) \leqslant C2^{N(1/p-\alpha)} \sum_{j=s+2}^{N+1} \sum_{i=j-s-2}^{j-2} 2^{(\alpha-1)i} 2^{j-s} 2^{i-N} \leqslant C2^{N/p} 2^{-s}.$$

Furthermore,

$$\begin{aligned}
&\sum_{l,l \text{ is even}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=0}^{j-s-3} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\leqslant C \sum_{l,l \text{ is even}} \left(\sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} 1 + \sum_{i=1}^{\gamma_l-\gamma_{l-1}+1} 2^{\alpha i} \right) 2^{-N} \leqslant C2^{-N} \\
&\quad \times \sum_{l,l \text{ is even}} (2^{\alpha(\gamma_l-\gamma_{l-1})} + \gamma_{l+1} - \gamma_l)
\end{aligned}$$

and analogously

$$\begin{aligned}
&\sum_{l,l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=0}^{j-s-3} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\leqslant C \sum_{l,l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} \sum_{i=j-\gamma_l+2}^{j-\gamma_l+\beta_j+2} 2^{(\alpha-1)i} 2^{i-N} 2^{j-\gamma_l+1},
\end{aligned}$$

where $\beta_j := 0$ when $\gamma_{l-1} \neq \gamma_l - 1$ and $\beta_j := \gamma_{l-1} - \gamma_{l-2}$ otherwise. Thus

$$\begin{aligned}
\sum_{l,l \text{ is odd}} \cdots &\leqslant C2^{-N} \sum_{l,l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} 2^{j-\gamma_l} 2^{\alpha(j-\gamma_l+\beta_j)} \\
&\leqslant C2^{-N} \sum_{l,l \text{ is odd}} \sum_{j=\gamma_l+2}^{\gamma_{l+1}+1} 2^{(\alpha+1)(\gamma_{l+1}-\gamma_l)}.
\end{aligned}$$

These estimations lead to

$$C(x) \leqslant C2^{N(1/p-\alpha-1)} \sum_l 2^{(\alpha+1)(\gamma_{l+1}-\gamma_l)}.$$

Hence, we have shown that for $x \in A_{s\gamma}$

$$\delta_{nN}(x) \leqslant C2^{N(1/p-\alpha-1)} \left(2^{(\alpha+1)s} + \sum_l 2^{(\alpha+1)(\gamma_{l+1}-\gamma_l)} \right) + C2^{N/p} 2^{-s} \chi_{I_N(e_s)}. \tag{7}$$

The same estimation is evidently true also for $\sup_{n \geqslant 2^N} \delta_{nN}(x)$.

Now, $\Delta_{nN}(x)$ will be investigated for $x \in I_s \backslash I_{s+1}$ $(s = 0, \ldots, N-1)$. To this end let G_{sl} $(l = s, \ldots, N-2)$ be the set of all $x \in I_s \backslash I_{s+1}$ such that $x_l = 1$ and $x_{l+1} = \cdots = x_{N-1} = 0$. It is clear that the measure of G_{sl} is 2^{l-N-s} and for $x \in G_{sl}$

$$\begin{aligned}
\Delta_{nN}(x) &\leqslant Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \left(\sum_{i=0}^{j-s-2} 2^{\alpha i} \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t))\, dt \right. \\
&\quad \left. + \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \right) \\
&\leqslant Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \left(\sum_{i=0}^{j-N-1} 2^{\alpha i}2^{-N} + \sum_{i=j-N}^{j-l-2} 2^i 2^{\alpha i} 2^{1-j} \right. \\
&\quad \left. + \sum_{i=0}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \right) \\
&\leqslant C(2^{N(1/p-\alpha-1)} + 2^{N/p}2^{-(\alpha+1)l}) \\
&\quad + Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \left(\sum_{i=0}^{j-N-1} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m 2^{-N} \right. \\
&\quad \left. + \sum_{i=j-N}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \right) \\
&\leqslant C(2^{N(1/p-\alpha-1)} + 2^{N/p}2^{-(\alpha+1)l}) \\
&\quad + Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \sum_{i=j-N}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\leqslant C(2^{N(1/p-\alpha-1)} + 2^{N/p}2^{-(\alpha+1)l}) \\
&\quad + Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \sum_{i=j-N}^{j-s-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&\quad + Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \sum_{i=j-s-1}^{j-2} 2^{(\alpha-1)i} \sum_{m=0}^{i-1} 2^m \int_{I_N} D_{2^i}(\tau_{j-1}(x \dot{+} t) \dot{+} e_m)\, dt \\
&=: C(2^{N(1/p-\alpha-1)} + 2^{N/p}2^{-(\alpha+1)l}) + E(x) + F(x).
\end{aligned}$$

We observe that $x \notin I_N(e_s)$ implies $F(x) = 0$ while in the case $x \in I_N(e_s)$ it follows that

$$\begin{aligned}
F(x) &\leqslant Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \sum_{i=j-s-1}^{j-2} 2^{(\alpha-1)i}2^{j-s-1}2^i 2^{-j} \\
&\leqslant Cn^{-\alpha}2^{N/p} \sum_{j=N+2}^{n_1} \sum_{i=j-s-1}^{j-2} 2^{\alpha i} \leqslant C2^{N/p}2^{-s}.
\end{aligned}$$

Furthermore, for $x \in G_{sl}$ $(l = s, \ldots, N-2)$ it can be written that

$$\begin{aligned} E(x) &\leqslant Cn^{-\alpha}2^{N/p}\sum_{j=N+2}^{n_1}\left(\sum_{i=j-N}^{j-l-2}2^{(\alpha-1)i}\sum_{m=0}^{j-N-1}2^m 2^i 2^{-j}\right.\\ &\quad\left.+\sum_{i=j-l-1}^{j-s-2}2^{(\alpha-1)i}2^{j-l-2}\int_{I_N}D_{2^i}(\tau_{j-1}(x\dot{+}t)\dot{+}e_{j-l-1})\,dt\right)\\ &\leqslant C2^{N(1/p-1)}2^{-\alpha l}+Cn^{-\alpha}2^{N/p}\sum_{j=N+2}^{n_1}\sum_{i=j-l-1}^{j-k-2}2^{(\alpha-1)i}2^{j-l-2}2^i 2^{-j} \end{aligned}$$

when $x \in G_{sl}^k \coloneqq \{x \in G_{sl} : x_k = 1, x_{k+1} = \cdots = x_{l-1} = 0\}$ $(k = s, \ldots, l)$. We remark that the measure of this set is evidently 2^{k-N-s}. Therefore if $x \in G_{sl}^k$ then

$$E(x) \leqslant C(2^{N(1/p-1)}2^{-\alpha l} + 2^{N/p}2^{-l}2^{-\alpha k}).$$

Hence, we are ready to estimate $\Delta_{nN}(x)$ $(x \in G_{sl}^k,\ l = s, \ldots, N-2; k = s, \ldots, l)$:

$$\Delta_{nN}(x) \leqslant C(2^{N(1/p-\alpha-1)} + 2^{N/p}2^{-(\alpha+1)l} + 2^{N/p}2^{-s}\chi_{I_N(e_s)} + 2^{N/p}2^{-l}2^{-\alpha k}). \tag{8}$$

Here, the right-hand side does not depend on n and so this estimation holds also for $\sup_{n \geqslant 2^N}\Delta_{nN}(x)$. Therefore by (7) and (8) we get the next maximal inequality (see (5)):

$$\begin{aligned} &\int_{[0,1)\setminus I_N}\left(\sup_n|\sigma_{n1}^{\alpha}a|\right)^p\\ &\quad\leqslant\int_{[0,1)\setminus I_N}\left(\sup_n|\sigma_{n1}^{\alpha1}a|\right)^p+\int_{[0,1)\setminus I_N}\left(\sup_n|\sigma_{n1}^{\alpha2}a|\right)^p\\ &\quad=\int_{[0,1)\setminus I_N}\left(\sup_n|\sigma_{n1}^{\alpha2}a|\right)^p\\ &\quad\leqslant C\int_{[0,1)\setminus I_N}\left(\left(\sup_{n\geqslant 2^N}|\delta_{nN}|\right)^p+\left(\sup_{n\geqslant 2^N}|\Delta_{nN}|\right)^p\right)\\ &\quad=C\sum_{s=0}^{N-1}\int_{I_s\setminus I_{s+1}}(\cdots)=C\sum_{s=0}^{N-1}\sum_{\gamma}\int_{A_{s\gamma}}\left(\sup_{n\geqslant 2^N}|\delta_{nN}|\right)^p\\ &\qquad+C\sum_{s=0}^{N-1}\sum_{l=s}^{N-2}\sum_{k=s}^{l}\int_{G_{sl}^k}\left(\sup_{n\geqslant 2^N}|\Delta_{nN}|\right)^p \end{aligned}$$

$$\leqslant C\sum_{s=0}^{N-1}\int_{I_s\setminus I_{s+1}} 2^{N(1-(\alpha+1)p)}2^{(\alpha+1)ps} + C2^{N(1-(\alpha+1)p)}$$

$$\times \sum_{s=0}^{N-1}\sum_{\gamma}\int_{A_{s\gamma}}\sum_{l} 2^{(\alpha+1)p(\gamma_{l+1}-\gamma_l)}$$

$$+ C\sum_{s=0}^{N-1} 2^N 2^{-ps}\int_0^1 \chi_{I_N(e_s)} + C\sum_{s=0}^{N-1}\int_{I_s\setminus I_{s+1}} 2^{N(1-(\alpha+1)p)}$$

$$+ C\sum_{s=0}^{N-1}\sum_{l=s}^{N-2}\int_{G_{sl}} 2^N 2^{-(\alpha+1)pl} + C\sum_{s=0}^{N-1}\int_0^1 2^N 2^{-ps}\chi_{I_N(e_s)}$$

$$+ C\sum_{s=0}^{N-1}\sum_{l=s}^{N-2}\sum_{k=s}^{l}\int_{G_{sl}^k}(2^{N(1-p)}2^{-\alpha pl} + 2^N 2^{-pl}2^{-\alpha pk})$$

$$\leqslant C2^{N(1-(\alpha+1)p)}\sum_{s=0}^{N-1} 2^{((\alpha+1)p-1)s} + C2^{N(1-(\alpha+1)p}$$

$$\times \sum_{s=0}^{N-1} 2^{-N}\sum_{\gamma}\sum_{l} 2^{(\alpha+1)p(\gamma_{l+1}-\gamma_l)}$$

$$+ C2^{N(1-(\alpha+1)p)}\sum_{s=0}^{N-1} 2^{-s} + C\sum_{s=0}^{N-1}\sum_{l=s}^{N-2} 2^{-s}2^{(1-(\alpha+1)p)l}$$

$$+ C\sum_{s=0}^{N-1}\sum_{l=s}^{N-2}\sum_{k=s}^{l}(2^{N(1-p)}2^{-\alpha pl} + 2^N 2^{-pl}2^{-\alpha pk})2^{k-N-s} + C$$

$$\leqslant C + C2^{N(1-(\alpha+1)p)}\sum_{s=0}^{N-1} 2^{-N}2^{(\alpha+1)p(N-s)}$$

$$+ C2^{-pN}\sum_{s=0}^{N-1} 2^{-s}\sum_{l=s}^{N-2} 2^{-\alpha pl}\sum_{k=s}^{l} 2^k$$

$$+ C\sum_{s=0}^{N-1} 2^{-s}\sum_{l=s}^{N-2} 2^{-pl}\sum_{k=s}^{l} 2^{(1-\alpha p)k} \leqslant C.$$

(We used the fact that $\sum_{\gamma}\sum_{l} 2^{(\alpha+1)p(\gamma_{l+1}-\gamma_l)} \leqslant C2^{(\alpha+1)p(N-s)}$.)

To complete the proof of Theorem 2 we need to estimate $|\sigma_{n2}^{\alpha}(x)|$ for $\mathbf{N} \ni n > 2^N$, $x \in I_s\setminus I_{s+1}$ $(s = 0, \dots, N-1)$. Applying the decomposition of $\mathscr{K}_{n2}^{\alpha}$ (see the proof of Theorem 1) let $\sigma_{n2}^{\alpha}(x) = \sigma_{n2}^{\alpha 1}(x) + \sigma_{n2}^{\alpha 2}(x) + \sigma_{n2}^{\alpha 3}(x)$ where

$$\sigma_{n2}^{\alpha i}(x) := \int_{I_N} a(t)\mathscr{K}_{n2}^{\alpha i}(x \dot{+} t)\,dt \quad (i = 1, 2, 3).$$

Then

$$|\sigma_{n2}^{\alpha 1}(x)| \leqslant \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{q} A_{n^{(k-1)}-1}^{\alpha} \int_{I_N} |a(t)| D_{2^{n_k}}(\tau_{n_1}(x\dot{+}t))\, dt$$

$$\leqslant Cn^{-\alpha} 2^{N/p} \sum_{k=2}^{q} (n^{(k-1)})^{\alpha} \int_{I_N} D_{2^{n_k}}(\tau_{n_1}(x\dot{+}t))\, dt$$

$$\leqslant Cn^{-\alpha} 2^{N/p} \sum_{k=2}^{q} 2^{\alpha n_k} \int_{I_N} D_{2^{n_k}}(\tau_{n_1}(x\dot{+}t))\, dt$$

$$= Cn^{-\alpha} 2^{N/p} \left(\sum_{k=2, n_k \geqslant n_1 - s} \cdots + \sum_{k=2, n_k \leqslant n_1 - N} \cdots + \sum_{k=2, n_1 - N < n_k < n_1 - s} \cdots \right)$$

$$=: Cn^{-\alpha} 2^{N/p} \left(\sum_1 + \sum_2 + \sum_3 \right).$$

By (1) it follows immediately that $D_{2^{n_k}}(\tau_{n_1}(x\dot{+}t)) = 0$ $(n_k \geqslant n_1 - s)$ for all $t \in I_N$, i.e. $\sum_1 = 0$. Furthermore, it is not hard to see that

$$\sum_2 \leqslant Cn^{-\alpha}\, 2^{N/p} \sum_{k=2, n_k \leqslant n_1 - N} 2^{\alpha n_k}$$

and for $x \in G_{sl}$ $(l = s, \ldots, N-1)$

$$\sum_3 \leqslant Cn^{-\alpha} 2^{N/p} \sum_{k=2, n_1 - N < n_k \leqslant n_1 - l - 1} 2^{n_k - n_1} 2^{\alpha n_k} \leqslant C 2^{N/p} 2^{-(\alpha+1)l}.$$

Therefore, the next estimation holds for all $x \in G_{sl}$ $(l = s, \ldots, N-1)$:

$$\int_{[0,1)\setminus I_N} \left(\sup_n |\sigma_{n2}^{\alpha 1}(x)| \right)^p dx = \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \cdots$$

$$\leqslant C 2^{N(1-(\alpha+1)p)} \sum_{s=0}^{N-1} 2^{-s} + C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} \int_{G_{sl}} 2^N 2^{-(\alpha+1)pl}$$

$$\leqslant C 2^{N(1-(\alpha+1)p)} + C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} 2^N 2^{-(\alpha+1)pl} 2^{-s} 2^{l-N} \leqslant C.$$

To the investigation of $\sigma_{n2}^{\alpha 3}(x)$ we apply the estimation (see (2))

$$|\mathscr{K}_{n2}^{\alpha 3}(x)| \leqslant Cn^{-\alpha} \sum_{k=2}^{q} \sum_{j=1}^{2^{n_k}-2} |A_{n^{(k)}+j+1}^{\alpha-2}|\, j |K_j(\tau_{n_1}(x)|$$

$$\leqslant Cn^{-\alpha} \sum_{k=2}^{q} \sum_{l=0}^{n_k - 1} \sum_{j=2^l}^{2^{l+1}-1} |A_{n^{(k)}+j+1}^{\alpha-2}|$$

$$\times \sum_{m=0}^{l} 2^m \sum_{i=m}^{l} (D_{2^i}(\tau_{n_1}(x)) + D_{2^i}(\tau_{n_1}(x)\dot{+}e_m))$$

$$\leqslant Cn^{-\alpha}\sum_{k=2}^{q}\sum_{l=0}^{n_k-1}\sum_{j=2^l}^{2^{l+1}-1}(n^{(k)}+j)^{\alpha-2}\left(\sum_{i=0}^{l}2^iD_{2^i}(\tau_{n_1}(x))\right.$$
$$\left.+\sum_{i=0}^{l}\sum_{m=0}^{i-1}2^mD_{2^i}(\tau_{n_1}(x)\dot{+}e_m)\right).$$

We remark that

$$\sum_{j=2^l}^{2^{l+1}-1}(n^{(k)}+j)^{\alpha-2}\leqslant C\sum_{j=2^l}^{2^{l+1}-1}j^{\alpha-2}\leqslant C2^{(\alpha-1)l}$$

so for every $x\in I_s\backslash I_{s+1}$ $(s=0,\dots,N-1)$

$$|\sigma_{n2}^{\alpha3}(x)|\leqslant C2^{N/p}n^{-\alpha}\sum_{k=2}^{q}\sum_{l=0}^{n_k-1}2^{(\alpha-1)l}\sum_{i=0}^{l}2^i\int_{I_N}D_{2^i}(\tau_{n_1}(x\dot{+}t))\,dt$$
$$+\sum_{i=0}^{l}2^i\sum_{m=0}^{i-1}2^i\int_{I_N}D_{2^i}(\tau_{n_1}(x\dot{+}t)\dot{+}e_m)\,dt$$
$$=C2^{N/p}n^{-\alpha}\sum_{k=2}^{q}\sum_{i=0}^{n_k-1}2^i\sum_{l=i}^{n_k-1}2^{(\alpha-1)l}\int_{I_N}D_{2^i}(\tau_{n_1}(x\dot{+}t))\,dt$$
$$+C2^{N/p}n^{-\alpha}\sum_{k=2}^{q}\sum_{i=0}^{n_k-1}\sum_{m=0}^{i-1}2^m\sum_{l=i}^{n_k-1}2^{(\alpha-1)l}\int_{I_N}D_{2^i}(\tau_{n_1}(x\dot{+}t)\dot{+}e_m)\,dt$$
$$=:G(x)+H(x).$$

Let us first investigated $G(x)$:

$$G(x)\leqslant C2^{N/p}n^{-\alpha}\left(\sum_{k=2,n_k\leqslant n_1-N+1}^{q}\sum_{i=0}^{n_k-1}2^i2^{(\alpha-1)i}2^{-N}\right.$$
$$+\sum_{k=2,n_k>n_1-N+1}^{q}\sum_{i=0}^{n_1-N}2^i2^{(\alpha-1)i}2^{-N}$$
$$\left.+\sum_{k=2,n_k>n_1-N+1}^{q}\sum_{i=n_1-N+1}^{n_k-1}2^i2^{(\alpha-1)i}\int_{I_N}D_{2^i}(\tau_{n_1}(x\dot{+}t))\,dt\right)$$

$$
\begin{aligned}
11 &\leqslant C2^{N(1/p-1)}n^{-\alpha}\left(\sum_{k=2,n_k\leqslant n_1-N+1}^{q} 2^{\alpha n_k} + \sum_{k=2,n_k>n_1-N+1}^{q} 2^{\alpha(n_1-N)}\right)\\
&\quad + C2^{N/p}n^{-\alpha}\sum_{k=2,n_k>n_1-N+1}^{q}\ \sum_{i=n_1-N+1}^{n_k-1} 2^{\alpha i}\int_{I_N} D_{2^i}(\tau_{n_1}(x\dot{+}t))\,dt\\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha}\sum_{k=2,n_k>n_1-N+1}^{q}\\
&\quad \times \sum_{i=n_1-N+1}^{n_k-1} 2^{\alpha i}\int_{I_N} D_{2^i}(\tau_{n_1}(x\dot{+}t))\,dt.
\end{aligned}
$$

If $l=s,\dots,N-1$ and $x\in G_{sl}$ then

$$
\begin{aligned}
G(x) &\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha}\sum_{k=2,n_k>n_1-l}^{q}\ \sum_{i=n_1-N+1}^{n_l-l-1} 2^{\alpha i}2^{i}2^{-n_1}\\
&\quad + C2^{N/p}n^{-\alpha}\sum_{k=2,n_1-l\geqslant n_k>n_1-N+1}^{q}\ \sum_{i=n_1-N+1}^{n_k-1} 2^{\alpha i}\,2^{i}2^{-n_1}\\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha}\sum_{k=2,n_k>n_1-l}^{q} 2^{(\alpha+1)(n_1-l)}2^{-n_1}\\
&\quad + C2^{N/p}n^{-\alpha}\sum_{k=2,n_1-l\geqslant n_k>n_1-N+1}^{q}\ \sum_{i=n_1-N+1}^{n_k-1} 2^{(\alpha+1)i}2^{-n_1}\\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}l2^{-(\alpha+1)l}\\
&\quad + C2^{N/p}n^{-\alpha}\sum_{k=2,n_1-l\geqslant n_k>n_1-N+1} 2^{(\alpha+1)n_k}2^{-n_1}\\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}l2^{-(\alpha+1)l}.
\end{aligned}
$$

Now, let $H(x)$ be estimated as follows:

$$
\begin{aligned}
H(x) &\leqslant C2^{N/p}n^{-\alpha}\sum_{k=2}^{q}\ \sum_{i=0}^{n_k-1}\ \sum_{m=0}^{i-1} 2^{m}2^{(\alpha-1)i}\int_{I_N} D_{2^i}(\tau_{n_1}(x\dot{+}t)\dot{+}e_m)\,dt\\
&= C2^{N/p}n^{-\alpha}\Bigg(\sum_{k=2,n_k\leqslant n_1-N+1}^{q}\ \sum_{i=0}^{n_k-1} 2^{\alpha i}2^{-N} + \sum_{k=2,n_k>n_1-N+1}^{q}\ \sum_{i=0}^{n_1-N} 2^{\alpha i}2^{-N}\\
&\quad + \sum_{k=2,n_k>n_1-N+1}^{q}\ \sum_{i=n_1-N+1}^{n_k-1}\ \sum_{m=0}^{i-1} 2^{m}2^{(\alpha-1)i}\int_{I_N} D_{2^i}(\tau_{n_1}(x\dot{+}t)\dot{+}e_m)\,dt\Bigg)\\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha}\sum_{k=2,n_k>n_1-N+1}^{q}\ \sum_{i=n_1-N+1}^{n_k-1}\\
&\quad \times \sum_{m=0}^{i-1} 2^{m}2^{(\alpha-1)i}\int_{I_N} D_{2^i}(\tau_{n_1}(x\dot{+}t)\dot{+}e_m)\,dt.
\end{aligned}
$$

Hence, from this it follows for $x \in G_{sl}^j$ $(l = s, \ldots, N-1; j = s, \ldots, l-1)$ that

$$\begin{aligned}
H(x) &\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha} \sum_{k=2, n_k > n_1 - N + 1}^{q} \\
&\quad \times \left(\sum_{i=n_1-N+1}^{n_{kl}-1} \sum_{m=0}^{i-1} 2^m 2^{(\alpha-1)i} 2^i 2^{-n_1} + \sum_{i=m-l}^{n_{kl}-1} 2^{n_1-l} 2^{(\alpha-1)i} 2^i 2^{-n_1} \right) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha} \sum_{k=2, n_k > n_1 - N + 1}^{q} \\
&\quad \times \left(\sum_{i=n_1-N+1}^{n_{kl}-1} 2^{\alpha i - N} + \sum_{i=n_1-l}^{n_{kl}-1} 2^{\alpha i - l} \right) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha} \sum_{k=2, n_k > n_1 - N + 1}^{q} \left(2^{-N} 2^{\alpha n_{kl}} + 2^{-l} 2^{\alpha n_{kl}} \right) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha} \left(\sum_{k=2, n_1 - l \geqslant n_k > n_1 - N + 1}^{q} 2^{-N} 2^{\alpha n_k} \right. \\
&\quad + \sum_{k=2, n_1 - j \geqslant n_k > n_1 - l}^{q} (2^{-N} 2^{\alpha(n_1 - l)} + 2^{-l} 2^{\alpha n_k}) \\
&\quad \left. + \sum_{k=2, n_k > n_1 - j}^{q} (2^{-N} 2^{\alpha(n_1-l)} + 2^{-l} 2^{\alpha(n_1 - j)}) \right) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}n^{-\alpha} (2^{-N} 2^{\alpha(n_1-l)} + (l-j) 2^{-N} 2^{\alpha(n_1-l)} \\
&\quad + 2^{-l} 2^{\alpha(n_1-j)} + j(2^{-N} 2^{\alpha(n_1-l)} + 2^{-l} 2^{\alpha(n_1-j)})) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N/p}((l-j)2^{-N}2^{-\alpha l} + j2^{-N}2^{-\alpha l} + j2^{-l}2^{-\alpha j}) \\
&\leqslant CN2^{N(1/p-(\alpha+1))} + C2^{N(1/p-1)} l 2^{-\alpha l} + C2^{N/p} j 2^{-l} 2^{-\alpha j},
\end{aligned}$$

where $n_{kl} := \min\{n_1 - l, n_k\}$.

By means of the above estimations we get for $\sigma_{n2}^{\alpha 3} a$ the next maximal inequality (see (5)):

$$\begin{aligned}
\int_{[0,1)\setminus I_N} \left(\sup_n |\sigma_{n2}^{\alpha 3} a| \right)^p &\leqslant C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} \int_{G_{sl}} G^p + C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} \sum_{j=s}^{l-1} \int_{G_{sl}^j} H^p \\
&\leqslant C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} (N^p 2^{N(1-(\alpha+1)p)} + 2^N l^p 2^{-(\alpha+1)pl}) 2^l 2^{-N-s} \\
&\quad + C \sum_{s=0}^{N-1} \sum_{l=s}^{N-1} \sum_{j=s}^{l-1} (N^p 2^{N(1-(\alpha+1)p)} \\
&\quad + 2^{N(1-p)} l^p 2^{-\alpha pl} + 2^N j^p 2^{-pl} 2^{-\alpha pj}) 2^j 2^{-N-s}
\end{aligned}$$

$$\leqslant CN^p 2^{N(1-(\alpha+1)p)} + C\sum_{l=1}^{\infty} l^p 2^{l(1-(\alpha+1)p)}$$
$$+ C\sum_{s=0}^{N-1} 2^{-s} \sum_{l=s}^{N-1} \sum_{j=s}^{l-1} 2^{-pN} l^p 2^{-\alpha pl} 2^j$$
$$+ C\sum_{s=0}^{N-1} 2^{-s} \sum_{l=s}^{N-1} \sum_{j=s}^{l-1} j^p 2^{j(1-\alpha p)} 2^{-pl}$$
$$\leqslant C + C2^{-pN} \sum_{l=1}^{N} l^p 2^{l(1-\alpha p)} + C\sum_{l=1}^{\infty} l^p 2^{l(1-(\alpha+1)p)}$$
$$\leqslant C + CN^p 2^{N(1-(\alpha+1)p)} \leqslant C. \tag{9}$$

Finally, we show that the estimation (9) remains true for $\sup_n |\sigma_{n2}^{\alpha 2} a|$ instead of $\sup_n |\sigma_{n2}^{\alpha 3} a|$. Indeed, by (2) it follows for $\mathscr{K}_{n2}^{\alpha 2}$ that

$$|\mathscr{K}_{n2}^{\alpha 2}(x)| \leqslant Cn^{-\alpha} \sum_{k=2}^{q} 2^{(\alpha-1)n_k} 2^{n_k-1} |K_{2^{n_k}-1}(\tau_{n_1}(x))|$$
$$\leqslant Cn^{-\alpha} \sum_{k=2}^{q} 2^{(\alpha-1)n_k} \sum_{m=0}^{n_k-1} 2^m \sum_{i=m}^{n_k-1} (D_{2^i}(\tau_{n_1}(x)) + D_{2^i}(\tau_{n_1}(x)\dot{+}e_m)).$$

Taking into consideration the estimation with respect to $\mathscr{K}_{n2}^{\alpha 3}$ and the inequality

$$\sum_{j=2^{n_k-1}}^{2^{n_k}-1} |A_{n^{(k)}+j+1}^{\alpha-2}| \leqslant C \sum_{j=2^{n_k-1}}^{2^{n_k}-1} (n^{(k)}+j+1)^{\alpha-2}$$
$$\leqslant C \int_{2^{n_k-1}}^{2^{n_k}-1} (n^{(k)}+x)^{\alpha-2}\, dx \leqslant C2^{(\alpha-1)n_k}$$

we get

$$\int_{[0,1)\setminus I_N} \left(\sup_n |\sigma_{n2}^{\alpha 2} a|\right)^p \leqslant C \int_{[0,1)\setminus I_N} (G^p + H^p).$$

This proves Theorem 2. □

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